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REFLECTION AND REFRACTION OF PLANE LONGITUDINAL WAVES AT THE INTERFACE OF A LIQUID AND AN ANOMALOUS ANISOTROPIC MEDIUM†

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The plane problem of the reflection and refraction of plane longitudinal waves at the interface of a liquid and a solid anisotropic half-space with elasticity constants which satisfy the condition $N = (a-d)b - c^2 < 0$, is investigated. The expression of the solutions of the problem in terms of the inverse apparent velocities of the waves and the unique determination on a Riemann surface enables a detailed analytical investigation to be made of the kinematic behaviour of the wave processes in question for different ratios of the elasticity constants of the contacting media. It is established that for certain angles of incidence the longitudinal waves excite two refracted quasi-transverse waves with different normal velocities and angles of refraction. This feature is directly related to the existence of acute-angled edges on the fronts of the quasi-transverse waves from a point source when $N < 0$.

In [1, 2], using Smirnov's and Sobolev's method, applied for the first time to Sveklo anisotropic media [3], we investigated the plane problem of the reflection and refraction of plane longitudinal waves at the interface of a liquid and a solid anisotropic half-space with four elasticity constants satisfying the condition $N > 0$. In this paper we extend the investigation of this problem to anomalous media satisfying the condition $N < 0$. In these media, the wave processes considered behave in a more complex way and require a special approach.

1. PLANE WAVES IN ANISOTROPIC MEDIA

Plane waves in an anisotropic medium with four elasticity constants can be expressed by the functions [2]

$$u_k = u(\Omega_k^+),\ v_k = v(\Omega_k^+),\ \Omega_k^\pm = t + \theta x \pm \lambda_k y \tag{1.1}$$

where

$$\lambda_k = \{H + (-1)^k [H^2 - (a/b)(1/a - \theta^2(1/d - \theta^2))]^{1/2}\}^{1/2} \tag{1.2}$$
$$H = [(b+d) - (ab + d^2 - c^2)\theta^2]/(2bd)$$

The functions are subject to the conditions

$$-u(\Omega_k^+)/(c\theta\lambda_k) = v(\Omega_k^+)/p_k = w(\Omega_k^+)$$
$$p_k = a\theta^2 + d\lambda_k^2 - 1 \tag{1.3}$$

The function w is an arbitrary continuous doubly differential function if the coefficients of w for variable quantities are real. If some of these coefficients in some region of space x, y, t are complex quantities, w is taken to be an analytic function in this region.

The normal velocities b_k and angles α_k, formed by the normals to the wave fronts and the y axis, are given by the expressions

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$$b_k = (\theta^2 + \lambda_k^2)^{-1/2}, \quad \operatorname{tg}\alpha_k = \theta/\lambda_k \quad (k = 1, 2) \tag{1.4}$$

The functions λ_1 and λ_2, represented by expressions (1.2), are branches of the algebraic function λ, uniquely defined on a Riemann surface, the form of which depends on the ratios of the elasticity constants.

The branching points for the inner radicals of (1.2) are the points [4]

$$\theta_i^0 = \pm[\{M \pm (4bdc^2[c^2 - (a-d)(b-d)])^{1/2}\}/(K_1K_2)]^{1/2} \tag{1.5}$$

$$K_1 = ab - (c-d)^2, \quad K_2 = ab - (c+d)^2$$

$$M = (b+d)[(a-d)(b-d) - c^2] - (a-d)(b-d)d$$

which may be complex, imaginary or real depending on the ratios of the elasticity constants.

When $N > 0$ the branching points for the outer radicals of (1.2) are the points $\theta_1 = \pm a^{-1/2}$ when $k = 1$ and the points $\theta_2 = \pm d^{-1/2}$ when $k = 2$. In this case the Riemann surface consists of the planes θ_1 and θ_2, respectively, with cuts $(-a^{-1/2}, +a^{-1/2})$ and $(-d^{-1/2}, +d^{-1/2})$. The planes are joined in a criss-cross manner along the corresponding cuts, connecting the branching points (1.5). If the branching points consist of two imaginary and two real points, the form of the Riemann surface is as derived previously ([5], Fig. 1).

On the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ and $(-d^{-1/2}, +d^{-1/2})$ of the planes θ_1 and θ_2, the functions λ_1 and λ_2 have real values, and the functions (1.1) express real plane waves: quasi-longitudinal for $k = 1$ and quasi-transverse for $k = 2$, propagating in any directions. Along the parts $(\pm a^{-1/2}, \pm\infty)$ and $(\pm d^{-1/2}, \pm\infty)$ of the real axes of the planes θ_1 and θ_2 the functions λ_1 and λ_2 take complex values, and the functions (1.1) express complex quasi-longitudinal and quasi-transverse waves.

Consequently, when $N > 0$, the quasi-longitudinal and quasi-transverse plane waves are expressed by the functions (1.1) for $k = 1$ and $k = 2$, defined on the real axes of the θ_1 and θ_2 planes. Hence, when solving the problem in question there was no need to use the Riemann surface [1, 2].

The situation is more complex when $N < 0$. The outer radical of the function λ_1 has four branching points: $\theta_1 = \pm a^{-1/2}$, $\theta_1 = \pm d^{-1/2}$; the outer radical of the function λ_2 has no branching points. Of the branching points (1.5) two are real and two are imaginary, where the condition $\theta_1^0 > d^{-1/2}$ is satisfied for the real points. The function λ_1 is single-valued in the θ_1 plane with cuts $(-a^{-1/2}, +a^{-1/2})$, $(\pm d^{-1/2}, \pm\theta_1^0)$ and $(\pm\theta_1^0, \pm\infty)$ along the real axis and $(\pm\theta_2^0, \pm i\infty)$ along the imaginary axis. The function λ_2 is single-valued in the θ_2 plane with cuts $(-\theta_1^0, +\theta_1^0)$ and $(\pm\theta_1^0, \pm\infty)$ along the real axis and $(\pm\theta_2^0, \pm i\infty)$ along the imaginary axis. The Riemann surface consists of the θ_1 and θ_2 planes joined in a criss-cross manner along the edges of the cuts $(\pm\theta_1^0, \pm\infty)$ and $(\pm\theta_2^0, \pm i\infty)$ (Fig. 1).

On the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ and $(\pm d^{-1/2}, \pm\theta_1^0)$ of the θ_1 plane and $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane, the functions λ_1 and λ_2 take real values, and the functions (1.1) express real waves. On the parts $(\pm a^{-1/2}, \pm d^{-1/2})$ of the θ_1 plane the function λ_1 has imaginary values and on the parts $(\pm\theta_1^0, \pm\infty)$ of the edges of the cuts of the θ_1 and θ_2 planes the functions λ_1 and λ_2 have complex values; the functions (1.1) express complex waves.

We will fix the functions λ_1 and λ_2 in the θ_1 and θ_2 planes so that they are positive when $\theta = i\beta$, where β is a fairly small positive quantity. Since the x and y axes coincide with the axes of elastic symmetry of

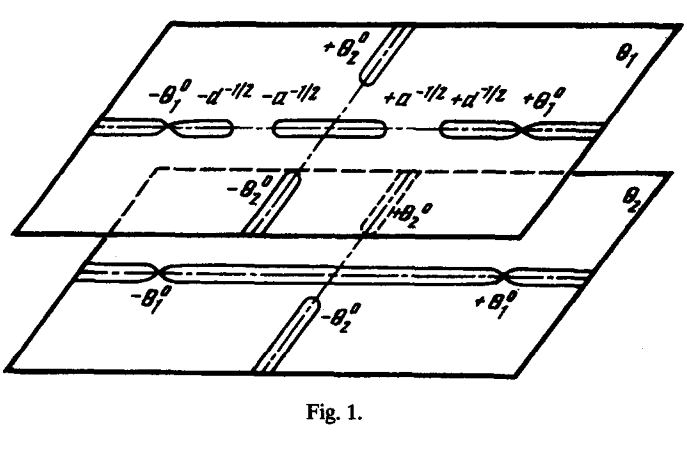

Fig. 1.

the medium, it is sufficient to investigate the wave propagation for positive real values of θ.

On the sections

$$0 \le \theta_1 \le a^{-1/2}, \quad 0 \le \theta_2 \le \theta_1^0 \tag{1.6}$$

of the upper edges of the cuts of the θ_1 and θ_2 planes, the functions λ_1 and λ_2 are positive, and the right-hand sides of the second formulae of (1.4) increase monotonically from zero to the values

$$\operatorname{tg}\alpha_1 = \infty, \quad \operatorname{tg}\alpha_2 = \theta_1^0/\lambda_2(\theta_1^0) \tag{1.7}$$

The functions (1.1) express real quasi-longitudinal and quasi-transverse waves propagating with continuously increasing angles α_1 and α_2 in the intervals

$$0 \le \alpha_1 \le \Pi/2, \quad 0 \le \alpha_2 \le \alpha_2^0 \quad (\alpha_2^0 < \Pi/2) \tag{1.8}$$

The normal velocities (1.4) of the waves on the sections (1.6) are continuous functions, having the following values on the boundaries of the sections

$$b_1(0) = b^{1/2}, \quad b_1(a^{-1/2}) = a^{1/2}, \quad b_2(0) = d^{1/2}, \quad b_2(\theta_1^0) < d^{1/2} \tag{1.9}$$

The nature of the change in the velocities depends on the values of the quantities [4, 6]

$$N_1 = a - d - c, \quad N_2 = b - d - c, \quad N_3 = (a-d)(b-d) - c^2 \tag{1.10}$$

Since $N_1 < 0$ when $N < 0$, the velocity of the quasi-longitudinal wave on the first part (1.6) decreases continuously and $b > a$ when $N_2 > 0$. If $N_2 < 0$, the velocity of the quasi-longitudinal wave inside this section has a maximum. The velocity of the quasi-transverse wave inside the second section of (1.6) has a minimum, since $N_3 > 0$ when $N < 0$.

The extremal points and the velocities and directions of propagation of the waves with these velocities have the values

$$\begin{gathered}
\theta_1^* = [(b-n)/(ab-n^2)]^{1/2}, \quad \theta_2^* = [(b+m)/(ab-m^2)]^{1/2} \\
b_1(\theta_1^*) = [(ab-n^2)/(a+b-2n)]^{1/2}, \quad b_2(\theta_2^*) = [(ab-m^2)/(a+b+2m)]^{1/2} \\
\operatorname{tg}\alpha_1^* = [(b-n)/(a-n)]^{1/2}, \quad \operatorname{tg}\alpha_2^* = [(b+m)/(a+m)]^{1/2} \\
m = c - d, \ n = c + d
\end{gathered} \tag{1.11}$$

When the branching point θ_1^0 passes round from the upper edge of the cut $(-\theta_1^0, +\theta_1^0)$ of the θ_2 plane of the Riemann surface to the lower edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane, the inner radical of the function λ_2 changes its sign from plus to minus, and the function λ_2 takes the value λ_1. The solutions (1.1)–(1.4), which express a quasi-transverse wave when $k = 2$, change to the solutions when $k = 1$, which are real plane waves defined on the lower edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane. These solutions have the same values at the branching point θ_1^0.

On the section

$$d^{-1/2} \le \theta_1 \le \theta_1^0 \tag{1.12}$$

of the lower edge of the cut of the θ_1 plane, the right-hand sides of (1.4) for $k = 1$ decrease monotonically for values on the boundaries

$$\begin{gathered}
\operatorname{tg}\alpha_1 = \infty, \quad \operatorname{tg}\alpha_1 = \theta_1^0/\lambda_1(\theta_1^0) = \theta_1^0/\lambda_2(\theta_1^0) \\
b_1(d^{-1/2}) = d^{1/2}, \quad b_1(\theta_1^0) = b_2(\theta_1^0)
\end{gathered}$$

Consequently, the functions (1.1)–(1.4) for $k = 1$ on the section (1.12) of the lower edge of the cut of the θ_1 plane express real quasi-transverse waves propagating in the directions

$$\Pi/2 \ge \alpha_1 \ge \alpha_1(\theta_1^0) = \alpha_2(\theta_1^0) \tag{1.13}$$

with normal velocities

$$d^{1/2} \ge b_1 \ge b_1(\theta_1^0) = b_2(\theta_1^0) \tag{1.14}$$

Graphs of the change in the normal velocities of the quasi-longitudinal and quasi-transverse waves as a function of the propagation direction are shown in Figs 2 and 3 by the continuous curves, while the dashed curves show the possible characteristic values of the normal velocities of the longitudinal waves in a liquid.

On the upper edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane of the Riemann surface the function λ_1 takes negative real values, and the functions (1.1) for $k = 1$ take the form

$$u_1 = u(\Omega_1^-),\ v_1 = v(\Omega_1^-) \tag{1.15}$$

$$u(\Omega_1^-)/(c\theta\lambda_1) = v(\Omega_1^-)/p_1 = w(\Omega_1^-)$$

where λ_1 has the value (1.2), and represents quasi-transverse waves. The quasi-transverse waves (1.1) for $k = 1$, defined on the lower edge of the cut $(+d^{-1/2}, +\theta_1^0)$ of the θ_1 plane and (1.15) are symmetrical with respect to x.

The direction of propagation of elastic vibrations is related to the motion of the energy in the deformed medium and is defined by the energy flux vector, which coincides with the radial (group) velocity vector [7]. Repeating the discussion given previously [8], we can express the projection of the energy flux vectors on to the coordinate axes for the case in question

$$S_x = -p\left\{\frac{\partial u}{\partial t}\left[a\frac{\partial u}{\partial x} + (c-d)\frac{\partial v}{\partial y}\right] + \frac{\partial v}{\partial t}\left[d\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\right\} \tag{1.16}$$

$$S_y = -\rho\left\{\frac{\partial u}{\partial t}\left[d\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial v}{\partial t}\left[(c-d)\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial y}\right]\right\}$$

Taking conditions (1.3) into account, we can express the quasi-longitudinal and quasi-transverse waves (1.1), defined in the sections $(0, +a^{-1/2})$ and $(0, +\theta^0{}_1)$ of the upper edges of the cuts of the θ_1 and θ_2 planes, by the functions

$$u_k = -c\theta\lambda_k w(\Omega_k^+),\ v_k = p_k w(\Omega_k^+) \tag{1.17}$$

Substituting (1.17) into (1.16) we obtain

$$S_{xk} = -\rho\theta p_k N_k [w'(\Omega_k^+)]^2,\ S_{yk} = -\rho\lambda_k p_k M_k [w'(\Omega_k^+)]^2 \tag{1.18}$$

$$N_k = 2ad\theta^2 + (ab + d^2 - c^2)\lambda_k^2 - (a+d)$$

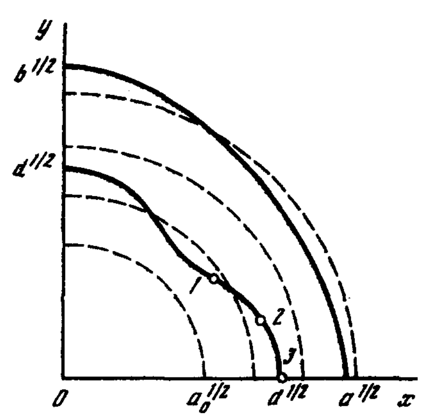

Fig. 2.

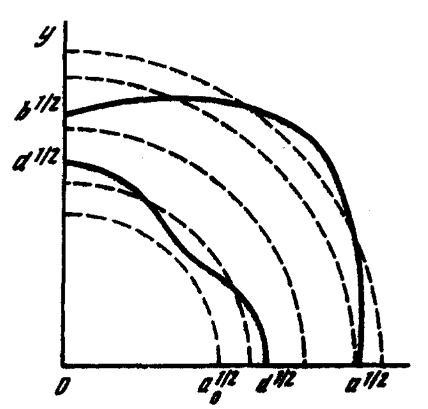

Fig. 3.

$$M_k = (ab + d^2 - c^2)\theta^2 + 2bd\lambda_k^2 - (b + d)$$

Since on the section $(0, a^{-1/2})$ $\lambda_1 > 0, p_1 < 0, M_1 < 0$, while on the section $(0, \theta^0{}_1)$ $\lambda_2 > 0, p_2 < 0$, $M_2 < 0$ the projections of the energy flux vectors onto the ordinate axis satisfy the conditions

$$S_{yk} = -\rho\lambda_k p_k M_k [w'(\Omega_k^+)]^2 < 0 \tag{1.19}$$

for $k = 1$ and $k = 2$ on the sections $(0, a^{-1/2})$ and $(0, \theta_1^0)$.

We can similarly determine the components of the energy flux vector of the quasi-transverse waves (1.15), defined in the section $(d^{-1/2}, \theta_1^0)$ of the upper edge of the cut of the θ_1 plane

$$S_{x1} = -\rho\theta p_1 N_1 [w'(\Omega_1^-)]^2; \quad S_{yl} = \rho\lambda_1 p_1 M_1 [w'(\Omega_1^-)]^2 \tag{1.20}$$

where N_1 and M_1 are the values of (1.18) for $k = 1$. Since on the section $(d^{-1/2}, \theta_1^0)$ $\lambda_1 > 0, p_1 > 0$, $M_1 < 0$, the projection of the energy flux vector of the waves (1.15) onto the ordinate axis satisfies the condition

$$S_{yl} = \rho\lambda_1 p_1 M_1 [w'(\Omega_1^-)]^2 < 0 \tag{1.21}$$

It follows from conditions (1.19) and (1.21) that the projections of the energy flux vectors and the radial velocities of the waves (1.1) and (1.15) onto the sections considered where they are determined, have negative values.

Henceforth, when solving the problem, the refracted quasi-longitudinal and quasi-periodic waves will be expressed using the functions (1.1) and (1.15), which ensure that the energy flows from the interface of the media $y = 0$ into the anisotropic medium $y < 0$.

2. REFLECTION AND REFRACTION OF LONGITUDINAL WAVES

A plane longitudinal wave [2]

$$u_0 = u(\Omega_0^+), \ v_0 = v(\Omega_0^+); \ \lambda_0 = (1/a_0 - \theta^2)^{1/2} \tag{2.1}$$

is incident from the liquid $y > 0$ onto an interface $y = 0$ with an anisotropic half-space.

The normal velocity and angles of incidence of the wave are given by the expressions

$$b_0 = a_0^{1/2} = (\mu_0/\rho_0)^{1/2}, \ \operatorname{tg}\alpha_0 = \theta/\lambda_0 \tag{2.2}$$

In the interval

$$0 \le \theta \le a_0^{-1/2} \tag{2.3}$$

the functions (2.1) represent a real wave with angles of incidence

$$0 \le \alpha_0 \le \Pi/2 \tag{2.4}$$

The qualitative picture of the reflection and refraction process depends on the ratios of the elasticity constants of the contacting media and the nature of the change in the normal velocities as a function of the direction of motion of the waves in the anisotropic medium, causing a variety of different combinations in the distribution of the velocities and directions of motion of the secondary waves and in the excitation of complex waves, depending on the angles of incidence of the primary waves. An investigation of these problems is of some theoretical and practical interest and reduces to considering three fundamental cases.

Case 1. The following condition is satisfied

$$a_0 > a > d, \text{ i.e. } a_0^{-1/2} < a^{-1/2} < d^{-1/2} \tag{2.5}$$

Since in the case of (2.5) the right boundaries of the sections (1.6) on the upper edges of the cuts of the θ_1 and θ_2 planes of the Riemann surface (Fig. 1) satisfy the condition $a_0^{-1/2} < a^{-1/2} < \theta^0{}_1$, in the section (2.3) the refracted quasi-longitudinal and quasi-transverse waves will be represented by the functions (1.1) with $k = 1$ and $k = 2$, respectively.

The reflected longitudinal and refracted quasi-longitudinal and quasi-transverse waves represent real waves and have the following expressions [2]

$$\begin{aligned} u_{00} &= (r_1/R)u(\Omega_0^-),\ v_{00} = -(r_1/R)v(\Omega_0^-) \\ u_{01} &= -(\lambda_1 c r_2/R)u(\Omega_1^+),\ v_{01} = (p_1 r_2/(\lambda_0 R))v(\Omega_1^+) \\ u_{02} &= -(\lambda_2 c r_3/R)u(\Omega_2^+),\ v_{02} = (p_2 r_3/(\lambda_0 R))v(\Omega_2^+) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} r_1 &= c(a/b)^{1/2}(\lambda_1 - \lambda_2)\{(\rho/\rho_0)[(ab)^{1/2}\xi_a + \\ &+(c-d)^2\theta^2\xi_d + ab\xi_a^2\xi_d]\lambda_0 - (ab)^{1/2}(\lambda_1+\lambda_2)\xi_a\}\xi_a \\ r_2 &= 2[a\xi_a^2 + (c-d)\lambda_2^2],\ r_3 = -2[a\xi_a^2 + (c-d)\lambda_1^2] \\ R &= c(a/b)^{1/2}(\lambda_1 - \lambda_2)\{(\rho/\rho_0)[(ab)^{1/2}\xi_a + (c-d)^2\theta^2\xi_d + ab\xi_a^2\xi_d]\lambda_0 + (ab)^{1/2}(\lambda_1+\lambda_2)\xi_a\}\xi_a \\ \xi_a &= (1/a - \theta^2)^{1/2},\ \xi_d = (1/d - \theta^2)^{1/2} \end{aligned} \tag{2.7}$$

$$(p_k = a\theta^2 + d\lambda_k - 1,\ k = 1, 2)$$

The normal velocities of the reflected and refracted waves and the angles of reflection and refraction are given by (2.2) and (1.4) and satisfy the sine law

$$\sin\alpha_0/b_0 = \sin\alpha_{00}/b_{00} = \sin\alpha_{01}/b_{01} = \sin\alpha_{02}/b_{02} = \theta \tag{2.8}$$

The functions (2.6) represent real waves for angles of incidence (2.4) of the longitudinal wave (2.1), defined in the interval (2.3). When the angle of incidence of the longitudinal wave increases the angles of reflection and refraction of waves (2.6) increase continuously, irrespective of how the normal velocities vary as a function of the direction of motion, since in sections (2.3) and (1.6) the right-hand sides of the second expressions of (2.2) and (1.7) increase continuously.

We will consider the distribution of the velocities and directions of motion of the primary and secondary waves as a function of the angle of incidence of the longitudinal wave when condition (2.5) is satisfied.

If $a_0^{1/2} > \max b_1$, it follows from Figs 2 and 3 and the sine law that for any angles of incidence of the longitudinal wave corresponding to the interval (2.3), the velocities and directions of motion satisfy the conditions

$$b_0 = b_{00} > b_{01} > b_{02},\qquad \alpha_0 = \alpha_{00} > \alpha_{01} > \alpha_{02} \tag{2.9}$$

If the velocity of the quasi-longitudinal wave has extremal values on the boundaries of the first section (1.6) (Fig. 2), then for $N < 0$ we have $b > a$. In this case, when the following condition is satisfied

$$\max b_1 = b^{1/2} > a_0^{1/2} > a^{1/2} \tag{2.10}$$

at a certain point θ_{11} in the section $(0, a^{-1/2})$, the velocity of the quasi-longitudinal wave will be equal to the velocity of the longitudinal wave.

If the point θ_{11} belongs to the section $(0, a_0^{-1/2})$, then for angles of incidence of the longitudinal wave defined in the section $(0, \theta_{11})$, the following conditions are satisfied

$$b_{01} > b_0 = b_{00} > b_{02},\qquad \alpha_{01} > \alpha_0 = \alpha_{00} > \alpha_{02} \tag{2.11}$$

and in the section $(\theta_{11}, a_0^{-1/2})$ conditions (2.9) are satisfied. When $\theta_{11} > a_0^{-1/2}$, condition (2.11) is satisfied in the section $(0, a_0^{-1/2})$.

If the velocity b_1 inside the first section of (1.6) has the greatest value (1.11), and the least value on the left boundary of the section (Fig. 3), then when (2.5) holds we may have the condition

$$\max b_1 = b_1(\theta_1^*) > a_0^{1/2} > a^{1/2} > \min b_1 = b^{1/2} \tag{2.12}$$

In the section $(0, \theta_1^*)$ the velocity b_1 increases continuously, while in the section $(\theta_1^*, a_0^{-1/2})$ it decreases continuously. At the points θ_{11} and θ_{12}, which belong to the sections $(0, \theta_1^*)$ and $(\theta_1^*, a^{-1/2})$, the velocity $b_1(\theta) = a_0^{1/2}$, and the right boundary of Section (2.3) is situated on the section $(\theta_{12}, a^{-1/2})$.

Consequently, for angles of incidence of the longitudinal wave given in (2.3), in the section $(0, \theta_{11})$ conditions (2.9) are satisfied, in the section $(\theta_{11}, \theta_{12})$ conditions (2.11) are satisfied and in the section $(\theta_{12}, a_0^{-1/2})$ conditions (2.9) are satisfied. If $(\theta_{12}, a_0^{-1/2})$, then in the section $(0, \theta_{11})$ conditions (2.9) are satisfied, while in the section $(\theta_{11}, a_0^{-1/2})$ conditions (2.11) are satisfied.

If the way in which the normal velocities vary differs from that shown in the graph in Fig. 3, only in the sense that $b^{1/2} > a^{1/2}$, then when (2.5) holds the following conditions may be satisfied

$$\max b_1 = b_1(\theta_1^*) > a_0^{1/2} > b^{1/2} > \min b_1 = a^{1/2} \tag{2.13}$$

$$\max b_1 = b_1(\theta_1^*) > b^{1/2} > a_0^{1/2} > \min b_1 = a^{1/2} \tag{2.14}$$

In this case, when (2.13) is satisfied, conditions (2.9) are satisfied in the sections $(0, \theta_{11})$ and $(\theta_{12}, a_0^{-1/2})$, and conditions (2.11) are satisfied in the section $(\theta_{11}, \theta_{12})$. If $(\theta_{12}, a_0^{-1/2})$, conditions (2.9) are satisfied in the section $(0, \theta_{11})$ and conditions (2.11) are satisfied in the section $(\theta_{11}, a_0^{-1/2})$.

When condition (2.14) $b_1(\theta) = a_0^{1/2}$ is satisfied at the point θ_{11} in the section $(\theta_1^*, a^{-1/2})$, the right boundary of the interval (2.3) belongs to the section $(\theta_{11}, a^{-1/2})$. Conditions (2.11) correspond to angles of incidence of the longitudinal wave in the interval (2.3) in the section $(0, \theta_{11})$, and conditions (2.9) in the section $(\theta_{11}, a_0^{-1/2})$. When $(\theta_{11} > a_0^{-1/2})$, conditions (2.11) are satisfied in the section (2.3).

Case 2. When

$$a > a_0 > d, \quad \text{i.e.} \quad a^{-1/2} < a_0^{-1/2} < d^{-1/2} \tag{2.15}$$

the right boundaries of sections (1.6) on the upper edges of the cuts of the θ_1 and θ_2 planes of the Riemann surface (Fig. 1) satisfy the condition

$$a^{-1/2} < a_0^{-1/2} < \theta_1^\circ$$

In the range (2.3), in which the incident longitudinal wave (2.1) is defined, the quasi-longitudinal and quasi-transverse waves are expressed by the functions (1.1) with $k = 1$ and $k = 2$, respectively.

The functions λ_1 and λ_2 have real values in the section $(0, a^{-1/2})$ of the range (2.3). The solution of the problem is given by the functions (2.1) and (2.6), which represent real waves.

If $b > a$ (Figs 2 and 3), the velocities and directions of motion of waves (2.1) and (2.6) satisfy conditions (2.11) in the section $(0, a^{-1/2})$.

When $b < a$ (Fig. 3) the following condition may be satisfied

$$\max b_1 = b_1(\theta_1^*) > a^{1/2} > a_0^{1/2} > \min b_1 = b^{1/2}$$

The point θ_{11} in which $b_1(\theta) = a_0^{1/2}$ belongs to the section $(0, a^{-1/2})$. In the section $(0, \theta_{11})$ conditions (2.9) hold for the velocities and directions of motion of the waves, while conditions (2.11) hold in the section $(\theta_{11}, a^{-1/2})$.

When

$$\min b_1 > a_0^{1/2} > d^{1/2}$$

(Figs 2 and 3) conditions (2.11) hold in the section $(0, a^{-1/2})$.

The function λ_1 takes imaginary values in the section $(a^{-1/2}, a_0^{-1/2})$ of the range (2.3). The solution of the problem can be expressed by functions of a complex variable [2]

$$u_0 = \mathrm{Re}[u_1(\Omega_0^+)], \quad v_0 = \mathrm{Re}[v_1(\Omega_0^+)]$$
$$u_{00} = \mathrm{Re}[(r_1^* / R^*)u_1(\Omega_0^-)], \quad v_{00} = \mathrm{Re}[-(r_1^* / R)v_1(\Omega_0^-)]$$
$$u_{01} = \mathrm{Re}[(i\lambda_1^* c r_2^* / R^*)u_1(\Omega_1^*)], \quad v_{01} = \mathrm{Re}[(p_1 r_2^* / (\lambda_0 R^*))v_1(\Omega_1^*)] \tag{2.16}$$

$$u_{02} = \mathrm{Re}[-(\lambda_2 c r_3^* / R^*)u_1(\Omega_2^+)], \quad v_{02} = \mathrm{Re}[(p_2 r_3^* / (\lambda_0 R^*))v_1(\Omega_2^+)]$$
$$\Omega_1^* = t + \theta x - i\lambda_1^* y$$

The quantities r_i^* and R^* are given by (2.7) when

$$\xi_a = -i(\theta^2 - 1/a)^{1/2}, \quad \lambda_1 = -i\lambda_1^*$$
$$\lambda_1^* = \{-H + [H^2 - (1/a - \theta^2)(1/d - \theta^2)(a/b)]^{1/2}\}^{1/2} \tag{2.17}$$

The functions u_1 and v_1 are regular functions in the upper half-plane of the complex variable. The refracted quasi-longitudinal wave is a complex wave with an imaginary phase velocity in the direction of the y axis, while the remaining waves are real.

The following conditions are satisfied for the velocities and directions of motion of the real waves, defined in the section $(a^{-1/2}, a_0^{-1/2})$

$$b_0 = b_{00} > b_{02}, \qquad \alpha_0 = \alpha_{00} > \alpha_{02} \tag{2.18}$$

Case 3. Suppose the following condition is satisfied

$$a > d > a_0, \text{ i.e. } \quad a^{-1/2} < d^{-1/2} < a_0^{-1/2} \tag{2.19}$$

In sections (1.6) of the upper edges of the cuts of the θ_1 and θ_2 planes of the Riemann surface (Fig. 1), the functions (1.1) represent quasi-longitudinal and quasi-transverse waves when $k = 1$ and $k = 2$, respectively, propagating in directions (1.8). In section (1.12) of the upper edge of the cut of the θ_1 plane, the function (1.1) takes the values (1.15) when $k = 1$ and represents quasi-transverse waves propagating in the directions (1.13).

Consequently, the incident longitudinal wave (2.1), defined in the section $(d^{-1/2}, \theta_1^0)$, excites two refracted quasi-transverse waves.

In the section $(0, a^{-1/2})$ of the range $(0, a_0^{-1/2})$ the solution of the problem is expressed by the real functions (2.1) and (2.6). The refracted waves are quasi-longitudinal and quasi-transverse waves.

In the section $(a^{-1/2}, d^{-1/2})$ of the range $(0, a_0^{-1/2})$ the solution of the problem is expressed by functions of the complex variable (2.16). The refracted quasi-longitudinal wave is a complex wave and the remaining ones are real.

On changing to the section $(d^{-1/2}, \theta_1^\circ)$ of the upper edges of the cuts of the θ_1 and θ_2 planes of the Riemann surface, the functions (2.16) take real values. The functions u_{01} and v_{01} become real and represent a quasi-transverse refracted wave.

In the sections $(d^{-1/2}, a_0^{-1/2})$ when $a_0^{-1/2} \leqslant \theta_1^\circ$ the solution of the problem is expressed by real functions

$$u_0 = u(\Omega_0^+), \quad v_0 = v(\Omega_0^+)$$
$$u_{00} = (r_1^* / R^*)u(\Omega_0^-), \quad v_{00} = -(r_1^* / R^*)v(\Omega_1^-) \tag{2.20}$$
$$u_{01} = (c r_2^* \lambda_1 / R)u(\Omega_1^-), \quad v_{01} = (p_1 r_2^* / (\lambda_0 R^*))v(\Omega_1^-)$$
$$u_{02} = -(c r_3^* \lambda_2 / R^*)u(\Omega_2^+), \quad v_{02} = (p_2 r_3^* / (\lambda_0 R^*))v(\Omega_2^+)$$

The quantities r_i^* and R^* are given by (2.7) with λ_1 replaced by $-\lambda_1$ and

$$\xi_a = -i(\theta^2 - 1/a)^{1/2}, \quad \xi_d = -i(\theta^2 - 1/d)^{1/2}$$

and have real values.

In this case the functions u_{0k} and v_{0k} $(k = 1, 2)$ represent real refracted quasi-transverse waves having different normal velocities and angles of refraction. The normal velocities and entry and exit angles of the waves are given by (2.2) and (2.3), and the sine law (2.8) is satisfied. By (1.19) and (1.21) the radiation principle is satisfied and the refracted waves transfer energy from the interface $y = 0$ into the anisotropic medium $y < 0$.

When $a_0^{-1/2} \leq \theta_1^\circ$ in the section $(\theta_1^\circ, a_0^{-1/2})$ the angles of incidence of the longitudinal wave (2.1) exceed the critical angle with respect to the refracted quasi-transverse waves and the functions λ_1 and λ_2 take complex values. The solution of the problem is expressed by functions of the complex variable

$$\begin{aligned} &u_0 = \mathrm{Re}[u_1(\Omega_0^+)], \quad v_0 = \mathrm{Re}[v_1(\Omega_0^+)] \\ &u_{00} = \mathrm{Re}[(\tilde{r}_1/\tilde{R})u_1(\Omega_0^-)], \quad v_{00} = \mathrm{Re}[-(\tilde{r}_1/\tilde{R})v_1(\Omega_0^-)] \\ &u_{01} = \mathrm{Re}[(\tilde{\lambda}_1 c\tilde{r}_2/\tilde{R})u_1(\tilde{\Omega}_1^-)], \quad v_{01} = \mathrm{Re}[(\tilde{p}_1\tilde{r}_2/(\lambda_0\tilde{R}))v_1(\tilde{\Omega}_1^-)] \\ &u_{02} = \mathrm{Re}[-(\tilde{\lambda}_2 c\tilde{r}_3/\tilde{R})u_1(\tilde{\Omega}_2^+)], \quad v_{02} = \mathrm{Re}[(\tilde{p}_2\tilde{r}_3/(\lambda_0\tilde{R}))v_1(\tilde{\Omega}_2^+)] \\ &\tilde{\Omega}_k^\pm = t + \theta x \pm \tilde{\lambda}_k y \end{aligned} \tag{2.21}$$

The quantities $\tilde{p}_k$, $\tilde{r}_i$ and $\tilde{R}$ are given by (2.7) with

$$\begin{aligned} &\lambda_k = (-1)^k \tilde{\lambda}_k \\ &\tilde{\lambda}_k = \{H - (-1)^k i[(\theta^2 - 1/a)(\theta^2 - 1/d)(a/b) - H^2]^{1/2}\}^{1/2} \\ &\xi_a = -i(\theta^2 - 1/a)^{1/2}, \quad \xi_d = -i(\theta^2 - 1/d)^{1/2} \end{aligned}$$

The quasi-transverse refracted waves u_{0k}, v_{0k} $(k = 1, 2)$ are complex waves with complex phase velocities in the direction of the y axis.

We will investigate the distribution of the velocities and directions of motion of the primary and secondary waves in the section $(0, a_0^{-1/2})$ when condition (2.19) is satisfied.

In the graph showing the change in the normal velocities b_1 and b_2 as a function of the directions of propagation of the quasi-transverse waves (Fig. 2), the values of these velocities, defined at the boundaries of the sections $(d^{-1/2}, \theta_1^\circ)$ in the θ_1 and θ_2 planes of the Riemann surface (Fig. 1), are denoted by the small circles as follows: (1) is the velocity $b_2(d^{-1/2})$, (2) is the velocity $b_2(\theta_1^\circ) = b_1(\theta_1^\circ)$, and (3) is the velocity $b_1(d^{-1/2}) = d^{1/2}$.

Graphs of the change in these velocities as a function of θ in the sections $(d^{-1/2}, \theta_1^\circ)$ are shown in Fig. 4. In this section the velocity b_2 increases continuously, the velocity b_1 decreases continuously and they are equal when $\theta = \theta_1^\circ$.

It follows from Figs 2–4 that when

$$b_1(d^{-1/2}) = d^{1/2} > a_0^{1/2} > b_1(\theta_1^\circ) = b_2(\theta_1^\circ)$$

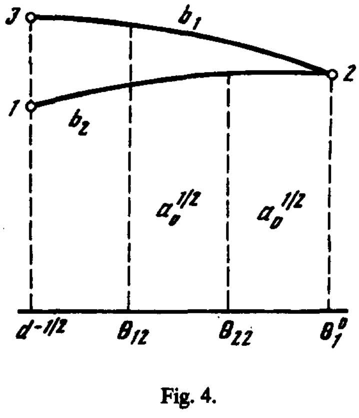

Fig. 4.

the velocities of the refracted quasi-transverse waves are equal to the velocities of the longitudinal wave at the points θ_{21} and θ_{12}, defined by the conditions $b_2(\theta_{21}) = a_0^{1/2}$ and $b_1(\theta_{12}) = a_0^{1/2}$, where the conditions $a_0^{-1/2} < \theta_1^\circ$ or $a_0^{-1/2} > \theta_1^\circ$ may be satisfied.

In this case, for angles of incidence of the longitudinal wave defined in the range $(0, a_0^{-1/2})$ for velocities and directions of motion of the primary and secondary waves the following conditions are satisfied: in the section $(0, \theta_{21})$, the conditions

$$b_{01} > b_{02} > b_0 = b_{00}, \qquad \alpha_{01} > \alpha_{02} > \alpha_0 = \alpha_{00} \tag{2.22}$$

in the section $(\theta_{21}, a^{-1/2})$, conditions (2.11), in which b_{01} are the velocities of the refracted quasi-longitudinal waves, in the section $(a^{-1/2}, d^{-1/2})$, conditions (2.18), in the section $(d^{-1/2}, \theta_{12})$, conditions (2.11), and in the sections $(\theta_{12}, a_0^{-1/2})$ when $a_0^{-1/2} < \theta_1^\circ$ and $(\theta_{12}, \theta_1^\circ)$ when $a_0^{-1/2} > \theta_1^\circ$, conditions (2.9), in which b_{10} are the velocities of the refracted quasi-transverse waves.

When

$$b_1(\theta_1^\circ) = b_2(\theta_1^\circ) > a_0^{1/2} > b_2(d^{-1/2})$$

the velocities of the refracted quasi-transverse waves are equal to the velocities of the longitudinal wave at the points θ_{21} and θ_{22}, defined by the equation $b_2(\theta) = a_0^{1/2}$, where the condition $a_0^{-1/2} > \theta_1^\circ$ is satisfied. For velocities and directions of motion of the primary and secondary waves in the range $(0, a_0^{-1/2})$ the following conditions are satisfied: in the section $(0, \theta_{21})$, conditions (2.22), in the section $(\theta_{21}, a^{-1/2})$, conditions (2.11), where b_{01} are the velocities of the refracted quasi-longitudinal waves, in the section $(a^{-1/2}, d^{-1/2})$, conditions (2.18), in the section $(d^{-1/2}, \theta_{22})$, conditions (2.11), and in the section $(\theta_{22}, \theta_1^\circ)$, conditions (2.22), where b_{01} are the velocity of the refracted quasi-transverse waves.

When

$$b_2(d^{-1/2}) > a_0^{1/2} > b_2(\theta_2^*) = \min b_2 \tag{2.23}$$

the points θ_{21} and θ_{22} belong to the sections $(0, \theta_2^*)$ and $(\theta_2^*, d^{-1/2})$. If $\theta_{22} < a^{-1/2}$, conditions (2.22) are satisfied in the sections $(0, \theta_{21})$ and $(\theta_{22}, a^{-1/2})$, and conditions (2.11) are satisfied in the section $(\theta_{21}, \theta_{22})$. In the section $(a^{-1/2}, d^{-1/2})$ we have the conditions

$$b_{02} > b_0 = b_{00}, \qquad \alpha_{02} > \alpha_0 = \alpha_{00} \tag{2.24}$$

When $(\theta_{22} > a^{-1/2})$ conditions (2.22) are satisfied in the section $(0, \theta_{21})$, while conditions (2.11) are satisfied in the section $(\theta_{21}, a^{-1/2})$. Here everywhere b_{01} is the velocity of the refracted quasi-longitudinal wave.

In the range $(a^{-1/2}, d^{-1/2})$ conditions (2.18) are satisfied for real waves in the section $(a^{-1/2}, \theta_{22})$ and conditions (2.24) are satisfied in the section $(\theta_{22}, d^{-1/2})$.

When (2.23) is satisfied, conditions (2.22) are satisfied in the section $(d^{-1/2}, \theta_1^\circ)$ where b_{01} is the velocity of the refracted quasi-transverse wave.

When

$$\min b_2 = b_2(\theta_2^*) > a_0^{1/2}$$

conditions (2.22) are satisfied in the sections $(0, a^{-1/2})$ and $(d^{-1/2}, \theta_1^\circ)$. The velocity b_{01} in the section $(0, a^{-1/2})$ is the velocity of the refracted quasi-longitudinal wave, while in the section $(d^{-1/2}, \theta_1^\circ)$ it is the velocity of the refracted quasi-transverse wave. Conditions (2.24) are satisfied in the section $(a^{-1/2}, d^{-1/2})$ for real waves.

In conclusion we note that a complete solution has thus been obtained for the problem of the reflection and refraction of longitudinal waves at the interface between a liquid and a solid anisotropic medium which satisfy the condition $N < 0$.

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